PQM - Addition of Angular Momentum

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Let $|J, M\rangle$ represent the combination of two subsystems that have total angular momentum j_1 and j_2 respectively. Here J is the net angular momentum of the combined system and M represents the amount of net angular momentum aligned along the $z-$axis. We want to express the states of the combined system (i.e. with varying values of J and M) in terms of the states of the subsystems.

The total angular momenta j_1 and j_2 of the individual subsystems do not change, so the maximum possible net angular momentum is $j := j_1 + j_2$. Begin with the highest weight state $|j, j\rangle$ where both subsystems are fully aligned along the $z-$axis:

$$|j, j\rangle = |j_1, j_1\rangle\, |j_2, j_2\rangle$$

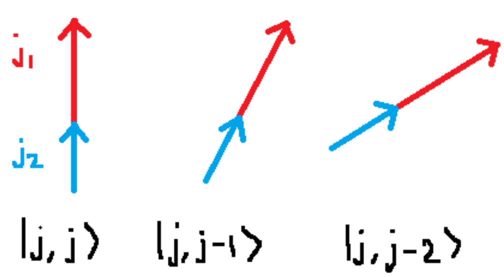
Since $-J \leq M \leq J$, we can obtain the states $|j, j-1\rangle, |j, j-2\rangle, \ldots$ by acting with the lowering operator

$$\mathbf{J}_- |J, M\rangle = \sqrt{J(J+1) - M(M-1)}\hbar\, |J, M-1\rangle \tag{1}$$

on both $|j, j\rangle$ as a whole, and on $|j_1, j_1\rangle\, |j_2, j_2\rangle$ with $\mathbf{J}_- = \mathbf{J}_{1z} \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes \mathbf{J}_{2z}$. Dividing through will eliminate the $\hbar$ factor, giving

$$|j, j-1\rangle = \sqrt{j_1/j}\, |j_1, j_1 - 1\rangle\, |j_2, j_2\rangle + \sqrt{j_2, j}\, |j_1, j_1\rangle\, |j_2, j_2 - 1\rangle\,.$$

Repeated use of the lowering operator gives all the states with $J = j$ but lowered values of M, corresponding to a rotation of the entire system where both j_1 and j_2 are still aligned in the same direction, but not along the $z-$axis.



This takes care of all the $J = j = j_1 + j_2$ states. To obtain the states of form $|J, M\rangle$ with $J < j$, first obtain $|j-1, j-1\rangle$ and then use the lowering operator as in (1) to get the remaining states for that value of J.

Recall that the action of $\mathbf{J}_z$ is

$$\mathbf{J}_z |J, M\rangle = M\hbar\, |J, M\rangle \tag{2}$$

so it makes sense to use this operator to see what happens to $|j-1,j-1\rangle$. Since we still have $j=j_1+j_2$, acting on this combined state gives

$$\mathbf{J}_z\,|j-1,j-1\rangle=(j_1+j_2-1)\hbar\,|j-1,j-1\rangle\,.$$

We don't actually know the form of $|j-1,j-1\rangle$ yet, but observing that in general

$$\mathbf{J}_z\,|j_\alpha,m_\alpha\rangle\,|j_\beta,m_\beta\rangle=(m_\alpha+m_\beta)\hbar\,|j_\alpha,m_\alpha\rangle\,|j_\beta,m_\beta\rangle$$

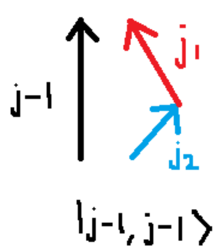
we can check that for a linear combination of combined states to be an eigenstate of $\mathbf{J}_z$, the corresponding m_α and m_β for each term must have the same sum. Hence, to agree with $\mathbf{J}_z\,|j-1,j-1\rangle$ we must have $m_\alpha+m_\beta=j_1+j_2-1$, so (m_α,m_β) is (j_1,j_2-1) or (j_1-1,j_2). Write the combined state as a linear combination

$$|j-1,j-1\rangle=a\,|j_1,j_1-1\rangle\,|j_2,j_2\rangle+b\,|j_1,j_1\rangle\,|j_2,j_2-1\rangle\,.$$

Then using normalisation $a^2+b^2=1$ and orthogonality $\langle j,j-1|j-1,j-1\rangle=0$, we find $a=\sqrt{j_2/j}$ and $b=-\sqrt{j_1/j}$.

Applying the lowering operator repeatedly gives the states $|j-1,j-2\rangle\,,|j-1,j-3\rangle\,,\ldots$. A similar construction using $\mathbf{J}_z$ allows $|j-2,j-2\rangle$ to be found in a similar way, obtaining the coefficients through normalisation and orthogonality.

The state $|j-1,j-1\rangle$ represents the case where j_1 and j_2 are imperfectly aligned with their net angular momentum being $j-1$, but where all of this net angular momentum is aligned along the $z-$axis. Again, applying the lowering operator rotates the system round in a semicircle.



How many states of the combined system are there in total? Intuitively, the lowest combined angular momentum occurs when j_1 and j_2 are anti-aligned, so is $|j_1-j_2|$. Without loss of generality, the combined J is in $\{j_1+j_2,j_1+j_2-1,\ldots,j_1-j_2\}$ so there are

$$\sum_{j=j_1-j_2}^{j_1+j_2}(2j+1)=(2j_1+1)(2j_2+1)$$

states in total.

We write the Clebsch-Gordan coefficients as

$$C_{j,m}(j_1,m_1;j_2,m_2)=\langle j,m|(|j_1,m_1\rangle\otimes|j_2,m_2\rangle)\rangle\,.$$