Complex Analysis
Definitions and Results (L1-16)

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Contents

1 Foundational Notions

1.1 Basic Definitions

We define the fundamental properties we need to do analysis on complex functions. Particularly important is the notion of a holomorphic function.

Definition (Continuity). Let $f : A \to \mathbb{C}$. Then f is continuous at $w \in A$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $z \in A$, $|z - w| < \delta \implies |f(z) - f(w)| < \varepsilon$.

Definition (Differentiability and holomorphicity). Let $U \subset C$ be open and let $f : U \to \mathbb{C}$. Then f is differentiable at $w \in U$ if the limit

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists in $\mathbb{C}$.
We say f is *holomorphic* at $w \in U$ if $\exists \varepsilon > 0$ s.t. $D(w, \varepsilon) \subset U$ and f is differentiable at every point in $D(w, \varepsilon)$. We say f is holomorphic in U if f is holomorphic at every point in U, or equivalently f is differentiable at every point in U.

1.2 Cauchy-Riemann Equations

We can identify complex functions with functions on $\mathbb{R}^2$ in the natural way. In what way is the complex differentiability related to differentiability of the corresponding function in $\mathbb{R}^2$?
We can write $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. It turns out that differentiability of f at $w = c + id \in U$ and differentiability of u and v at (c, d) are *not* equivalent! We also need the Cauchy-Riemann equations to hold.

Theorem (Cauchy-Riemann equations). The function $f = u + iv : U \to \mathbb{C}$ is differentiable at $w = c + id \in U$ if and only if $u, v : U \to \mathbb{R}$ are differentiable at $(c, d) \in U$ *and* u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

In this case, we get $f'(w) = u_x(c, d) + iv_x(c, d)$.

Corollary. Let $f = u + iv : U \to \mathbb{C}$. If u and v have continuous partial derivatives at $c, d \in U$ and satisfy Cauchy-Riemann equations at (c, d), then f is differentiable at $w = c + id$. In particular, if u and v are C^1 on U and satisfy Cauchy-Riemann in U, then f is holomorphic in U.

Theorem (Looman-Menchoff Theorem). If $f = u + iv$ is defined on an open set U and is continuous in U, and u and v satisfy Cauchy-Riemann in U, then f is holomorphic in U.

1.3 Power Series

We can prove standard properties of complex-value power series using the same methods as in IA Analysis I.

Theorem. Let $\sum_{n=0}^{\infty} c_n (z - a)^n$ be a power series with radius of convergence $R > 0$. Fix $a \in \mathbb{C}$ and define $f : D(a, R) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$. Then
(i) f is holomorphic on $D(a, R)$
(ii) The derived series $\sum_{n=1}^{\infty} n c_n (z - a)^{n-1}$ also has radius of convergence R and equals $f'(z)$
(iii) f has derivatives of all orders on $D(a, R)$ and $c_n = f^{(n)}(a)/n!$
(iv) if f vanishes on $D(a, \varepsilon)$ for some $\varepsilon > 0$ then $f = 0$ on $D(a, R)$.

1.4 Exponentials and Logarithms

Definition (Entire). If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, then f is entire.

Definition (Complex exponential function). The complex exponential function e^z is defined by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Definition (Logarithm). Let $z \in \mathbb{C}$. Then $w \in \mathbb{C}$ is a logarithm of z if $e^w = z$.

Definition (Branch of logarithm). Let $U \subset \mathbb{C}\backslash\{0\}$ be open. Then a branch of logarithm on U is a continuous function $\lambda : U \to \mathbb{C}$ with $e^{\lambda(z)} = z$ for each $z \in U$.
If λ is a branch of logarithm on U, then λ is automatically holomorphic in U with $\lambda'(z) = 1/z$.

Definition (Principal branch of logarithm). The principal branch of logarithm is the function $\text{Log} : U_1 = \mathbb{C}\backslash\{x \in \mathbb{R} : x \leqslant 0\} \to \mathbb{C}$ defined by

$$\text{Log}(z) = \log|z| + i \arg z$$

for $\arg z \in (-\pi, \pi)$.

2 Conformal Maps

2.1 Definitions and Properties

We find that some holomorphic functions have the property of being angle-preserving. These functions, called conformal maps, are useful for mapping between domains. They have useful properties such as preserving harmonicity of a function under composition, and can be used to solve Laplace's equation.

Definition (Conformal map). A function $f : U \to \mathbb{C}$ on an open set U is *conformal* at $w \in U$ if f is holomorphic and $f'(w) \neq 0$.

Definition (Conformal equivalence). Let U and $\tilde{U}$ be domains in $\mathbb{C}$. A map $f : U \to \tilde{U}$ is a *conformal equivalence* between U and $\tilde{U}$ if f is a bijective holomorphic map with $f'(z) \neq 0$ for all $z \in U$.

Proposition (Converse property). If f is a C^1-map on U, then the converse to the angle-preserving statement holds:
If for $w \in U$, f has the property that $(f \circ \gamma)'(0) \neq 0$ for all C^1 curves γ with $\gamma(0) = w$ and $\gamma'(0) \neq 0$, and if f is angle preserving at w, then $f'(w)$ exists and $f'(w) \neq 0$. (See the first example sheet).

2.2 Useful Conformal Maps

- We can show that all Möbius maps are conformal maps.
- Let $\mathbb{H}$ be the upper half-plane $\mathbb{H} = \{z : \text{Im}(z) > 0\} \cap \mathbb{C}$. Then $z \in \mathbb{H}$ if and only if $|z - i| < |z + i|$. Hence the conformal map

  $$z \mapsto \frac{z - i}{z + i}$$

  maps $\mathbb{H}$ onto the unit disc $D(0, 1)$.

3 Complex Integration

3.1 Definitions

We can define integration for complex functions in an analogous way to integration for real functions.

Definition (Integral). If $f : [a,b] \subset \mathbb{R} \to \mathbb{C}$ is a continuous complex function (or more generally, if $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Riemann integrable), we define

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}(f(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}(f(t))\,\mathrm{d}t.$$

Definition (Integral along curve). Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. Let $\gamma : [a,b] \to U$ be a C^1 curve. Then the integral of f along γ is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t.$$

This satisfies expected properties such as invariance under reparametrisation, linearity, additivity and inverse path.

Definition (Length of curve). Let $\gamma : [a,b] \to \mathbb{C}$ be a C^1-curve. Then the length of γ is defined by

$$\mathrm{length}(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t.$$

Definition (Sum of curves). If $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are curves with $\gamma_1(b) = \gamma_2(c)$, we define the sum of γ_1 and γ_2 to be $(\gamma_1 + \gamma_2) : [a, b+d-c] \to \mathbb{C}$ with

$$(\gamma_1+\gamma_2)(t) = \begin{cases} \gamma_1(t) & \text{for } a \leqslant t \leqslant b \\ \gamma_2(t-b+c) & \text{for } b \leqslant t \leqslant b+d-c. \end{cases}$$

3.2 Fundamental Theorem of Calculus

We can prove a result analogous to the fundamental theorem of calculus for complex functions.

Theorem (Fundamental Theorem of Calculus). Suppose $f : U \to \mathbb{C}$ is continuous and $U \subseteq \mathbb{C}$ is open. If there is a function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)\ \forall z \in U$, then for any curve $\gamma : [a,b] \to U$,

$$\int_\gamma f(z)\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a)).$$

It follows that if such an antiderivative F exists, then the integral of f over a closed curve is 0.
It turns out that we can actually prove a converse to this: if the integral over *every* closed curve vanishes, then the function has an antiderivative. This is a somewhat powerful condition.

Theorem (Converse to FTC). Let $U \subset \mathbb{C}$ be a domain. If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f(z)\,\mathrm{d}z = 0$ for *every* closed curve γ in U, then f has an antiderivative (i.e. $\exists$ holomorphic $F : U \to \mathbb{C}$ with $F'(z) = f(z)$ $\forall z \in U$).
(In this case, the antiderivative F is $F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z$, where γ_w is a path from a fixed point a_0 to w).

3.3 Cauchy's Theorem

We have just shown that if a *continuous* function has an antiderivative, then integrating the function over a closed curve gives zero. We may ask the question: in what domains does integrating a holomorphic function over a closed curve give zero? It turns out such domains are the *simply connected* domains, a result given by Cauchy's integral theorem.

Definition (Star-shaped domain). A domain U is star-shaped if there exists $a_0 \in U$ such that $\forall w \in U$, the straight line segment $[a_0, w] \subset U$.

Definition (Triangle in $\mathbb{C}$). A triangle T in $\mathbb{C}$ is the convex hull of three points $z_1, z_2, z_3 \in \mathbb{C}$. In this case

$$T = \{az_1 + bz_2 + cz_3 : 0 \leqslant a, b, c \leqslant 1, a + b + c = 1\}.$$

Corollary (check using FTC converse). If U is star-shaped, $f : U \to \mathbb{C}$ is continuous and $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for *any* triangle $T \subset U$, then f has an antiderivative in U.

Theorem (Cauchy's theorem for triangles). Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Then

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

for any triangle $T \subset U$.
To prove this we repeatedly subdivide a triangle T into four smaller triangles by joining the midpoints; then we can form a nested sequence $T = T_0, T_1, T_2, \ldots$ and letting $\eta(T_i) = \int_{\partial T_i} f(z)\,\mathrm{d}z$ we can bound $\frac{1}{4^n}|\eta(T_0)| \leqslant |\eta(T_n)|$ and working with lengths and using differentiability properties we can bound this above by something that goes to 0.

Theorem (Generalisation of Cauchy's theorem for triangles). Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. Let $S \subset U$ be a finite set and suppose f is holomorphic on $U \backslash S$. Then

$$\int_{\partial T} f(z)\,\mathrm{d}z = 0$$

for every triangle $T \subset U$.

Corollary (Convex Cauchy's theorem). Let $U \subset \mathbb{C}$ be convex (or a star domain). Let $f : U \to \mathbb{C}$ be continuous and holomorphic in $U \backslash S$ for some finite S. Then

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

for any closed curve γ in U.

3.4 Cauchy's Integral Formula

Suppose we know f is holomorphic inside a given disk. Then we can express the value of the function at any point inside the disk in terms of an integral over the boundary of the disk.

Theorem (Cauchy's Integral Formula for disk). Let $D = D(a, r)$ and $f : D \to \mathbb{C}$ be holomorphic. Then for any ρ with $0 \leqslant \rho \leqslant r$ and any $w \in D(a, \rho)$ we have

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z - w}\,\mathrm{d}z.$$

In particular, taking $w = a$ gives

$$f(a) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-a} \,\mathrm{d}z \implies f(a) = \int_0^1 f(a + \rho e^{2\pi i t}) \,\mathrm{d}t$$

which is called the mean value property for holomorphic functions.

3.5 Liouville's Theorem and the Fundamental Theorem of Algebra

Using Cauchy's integral formula for a disk, we can prove the following, very powerful result:

Theorem (Liouville's Theorem). If $f : \mathbb{C} \to \mathbb{C}$ is entire and bounded, then f is constant.
(More generally, if f is entire with sub-linear growth i.e. $\exists K \geqslant 0,\ \alpha < 1$ s.t. $|f(z)| \leqslant K(1 + |z|^\alpha)\ \forall z \in \mathbb{C}$, then f is constant).

Theorem (Fundamental Theorem of Algebra). Every non-constant polynomial with complex coefficients has a complex root.

3.6 Local Maximum Modulus Principle

Here we consider a function f holomorphic on a disk. If $|f|$ attains a maximum value on the disk, then f is constant.

Theorem (Local maximum modulus principle). If $f : D(a,R) \to \mathbb{C}$ is holomorphic with $|f(z)| \leqslant |f(a)|$ $\forall z \in D(a,R)$, then f is constant.

3.7 Taylor Series

Early on, we have seen that power series are a way to construct holomorphic functions on a disk. In fact, *every* holomorphic function on a disk arises this way.

Theorem (Taylor series). Let $f : D(a,R) \to \mathbb{C}$ be holomorphic. Then f has a convergent power series representation on $D(a,R)$. More precisely, there is a sequence of complex numbers $c_0, c_1, c_2, \ldots$ such that

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n$$

for all $w \in D(a,R)$, with coefficients

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z$$

for all $\rho \in (0,R)$.

It follows from this that if f is holomorphic on an open set $U \subset \mathbb{C}$, then f has derivatives of all orders in U which are *themselves* holomorphic on U.

Definition (Analytic). A function f is *analytic* at a point a if in a neighbourhood of a, f is given by a convergent power series about a.

For complex functions, f is analytic if and only if it is holomorphic. This is not true for real functions e.g. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-1/x^2}$, which has $f^{(n)}(0) = 0$ for all n, but is not analytic.

3.8 Morera's Theorem

This is a useful criterion for determining whether a continuous function is holomorphic.

Theorem (Morera's Theorem). Let $U \subset \mathbb{C}$ be open. If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all closed curves γ in U, then f is holomorphic in U.

3.9 Isolated Points

An isolated point of a set in $\mathbb{C}$ can be isolated from every other element of the set by surrounding it with a small enough disk.

Definition (Isolated point). Let $S \subset \mathbb{C}$. We say that $w \in S$ is an *isolated point* of S if there is $r > 0$ such that $S \cap D(w, r) = \{w\}$.

We can show that the zeros of a non-zero holomorphic function are isolated points. (This can be seen by taking out a factor $(z - a)^m$ from the power series of the function).

Theorem (Principle of isolated zeros). Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and not identically zero. Then there exists r with $0 < r < R$ such that $f(z) \neq 0$ whenever $0 < |z - a| < r$.

(This says that f is nonzero on a small enough disk around the centre of the domain, except perhaps at the centre point).

A number of useful consequences follow:

- If $f(a) = 0$ (with f not identically zero), then a is an isolated point of the zero set. It follows that there is no nonzero holomorphic function that vanishes on a line segment or a half disk.

- The zero set may have an accumulation point on the boundary of the domain of f.

3.10 Unique Continuation

Consider $f : D(a, r) \to \mathbb{C}$ holomorphic. By the Taylor series theorem, f is uniquely determined by its values in an arbitrarily small disk, because the coefficients in the expansion are determined by values on the disk. This notion of "extension" of a function to a larger domain can be generalised.
The following theorem asserts uniqueness of such a continuation, but *not* necessarily existence.

Theorem (Unique continuation for analytic functions). Let U and V be domains with $U \subset V$. If $g_1, g_2 : V \to \mathbb{C}$ are analytic functions and $g_1 = g_2$ on U, then $g_1 = g_2$ on V.

Corollary (Identity principle). Let $f, g : U \to \mathbb{C}$ be holomorphic in a domain U. If the set $S = \{z \in U : f(z) = g(z)\}$ contains a non-isolated point, then $f = g$ in U.

Corollary (Global maximum principle). Let U be open and bounded. Suppose $f : \bar{U} \to \mathbb{C}$ is continuous with f holomorphic in U. Then $|f|$ attains its maximum on $\partial U = \bar{U} \backslash U$.

3.11 Cauchy's Integral Formula for Derivatives

We in fact have an integral formula for an nth derivative of f that depends only on f itself, and not on any of its derivatives.

Theorem (CIF for derivatives). Let $f : D(a,R) \to \mathbb{C}$ be holomorphic. If $f^{(n)}$ denotes the nth derivative, then for any $\rho \in (0,R)$, $w \in D(a,\rho)$, we have

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{n+1}} \, \mathrm{d}z.$$

3.12 Cauchy Estimate

Using Cauchy's integral formula for derivatives, we obtain the following:

$$\sup_{z \in D(a,R/2)} |f^{(k)}(z)| \leqslant \frac{C}{R^k} \sup_{z \in D(a,R)} |f(z)|$$

where C is a constant depending only on k; in fact we can take $C = k!\, 2^{k+1}$. This allows us to bound the magnitude of the kth derivative on a disk of half the radius of the original.

3.13 Uniform Limits of Holomorphic Functions

We may want to see what we can say about the properties of a uniform limit of holomorphic functions, and see how they compare to the corresponding properties for continuous and differentiable real functions. What properties does a uniform limit preserve?

Definition (Local uniform convergence). Let $U \subset \mathbb{C}$ be open and $f_n : U \to \mathbb{C}$ be a sequence of functions. We say (f_n) converges *locally uniformly* on U if for each $a \in U$, there exists $r > 0$ such that r converges *uniformly* on $D(a,r)$.

Proposition. The sequence (f_n) converges locally uniformly on U if and only if (f_n) converges uniformly on each *compact* subset $K \subset U$.

Theorem (Uniform limits of holomorphic functions). Let $U \subset \mathbb{C}$ be open and $f_n : U \to \mathbb{C}$ be holomorphic for each n. If (f_n) converges locally uniformly on U to some $f : U \to \mathbb{C}$, then f is holomorphic. Moreover, $f_n' \to f'$ locally uniformly on U.

4 Complex Integration II

In this section, we want to (i) given a domain, characterise the closed curves in it for which Cauchy's theorem holds for all holomorphic functions, and (ii) use this to enlarge the class of domains for which Cauchy's theorem holds.

4.1 Winding Number

The winding number is a way to quantify "how many times" a curve winds around a point.

Definition (Winding number). Let $\gamma : [a,b] \to \mathbb{C}$ be a closed, piecewise C^1 curve with $w \notin \operatorname{img}(\gamma)$. Let $r : [a,b] \to \mathbb{R}$, $r(t) = |\gamma(t) - w|$. If there exists a continuous $\theta : [a,b] \to \mathbb{R}$ such that

$$\gamma(t) = w + r(t)e^{i\theta(t)}$$

then the *winding number* of γ about w is

$$I(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi}.$$

We can check this is well-defined and always gives an integer. In fact, for a well-behaved curve, we can show such a θ exists, and express the winding number in a nice integral form.

Lemma (Integral formula for winding number). If $w \in \mathbb{C}$, $\gamma : [a,b] \to \mathbb{C}\backslash\{w\}$ is a piecewise C^1 curve, then there exists a piecewise C^1 function $\theta : [a,b] \to \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$ where $r(t) = |\gamma(t) - w|$. Moreover if γ is closed, then

$$I(\gamma; w) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - w}.$$

Proposition. If $\gamma : [a,b] \to \mathbb{C}$ is a closed curve, then the function $w \mapsto I(\gamma; w)$ is continuous on $\mathbb{C}\backslash\mathrm{img}(\gamma)$. Hence (since $I \in \mathbb{Z}$), I is locally constant.

Proposition. (i) If $\gamma : [a,b] \to D(z_0, R)$ is a closed curve, then $I(\gamma; w) = 0$ for any $w \in \mathbb{C}\backslash D(z_0, R)$
(ii) If $\gamma : [a,b] \to \mathbb{C}$ is a closed curve, then there is a unique unbounded connected component Ω of $\mathbb{C}\backslash\gamma([a,b])$ and $I(\gamma; w) = 0$ for all $w \in \Omega$.

Lemma. Let $f : U \to \mathbb{C}$ be holomorphic and define $g : U \times U \to \mathbb{C}$ by

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z - w \end{cases}.$$

Then g is continuous, and γ is a closed curve in U, then $h(w) := \int_\gamma g(z,w)\,\mathrm{d}z$ is holomorphic on U.

4.2 Fubini's Theorem

This is a special case of a theorem that essentially allows us to interchange the order of integration.

Theorem (Fubini's theorem special case). If $\phi : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then the function $f_1 : s \mapsto \int_c^d \phi(s,t)\,\mathrm{d}t$ is continuous on $[a,b]$, the function $f_2 : t \mapsto \int_a^b \phi(s,t)\,\mathrm{d}s$ is continuous on $[c,d]$, and

$$\int_a^b \left(\int_c^d \phi(s,t)\,\mathrm{d}t\right)\mathrm{d}s = \int_c^d \left(\int_a^b \phi(s,t)\,\mathrm{d}s\right)\mathrm{d}t.$$

5 General Cauchy's Theorem

To generalise this statement, we need the concept of being homologous to zero. For a closed curve, this means its winding number around any point not on the curve is zero.

Definition (Homologous to zero). Let $U \subset \mathbb{C}$ be open. A closed curve $\gamma : [a,b] \to U$ is *homologous to zero* in U if $I(\gamma; w) = 0$ for every $w \in \mathbb{C}\backslash U$.

5.1 General Cauchy Theorem and Integral Formula

Theorem (General Cauchy theorem and CIF). Let U be a nonempty open subset of $\mathbb{C}$ and γ be a closed curve in U homologous to zero in U. Then
(i) For every holomorphic $f : U \to \mathbb{C}$ and every $w \in U\backslash\mathrm{img}(\gamma)$

$$I(\gamma; w)f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - w}\,\mathrm{d}z$$

(ii) For every holomorphic $f : U \to \mathbb{C}$

$$\int_\gamma f(z)\,\mathrm{d}z = 0.$$

Corollary. Let $U \subset \mathbb{C}$ be open and $\gamma_1, \gamma_2, \ldots, \gamma_n$ be closed curves in U such that $\sum_{j=1}^{n} I(\gamma_j; w) = 0$ for all $w \in \mathbb{C}\backslash U$. Then for any holomorphic $f : U \to \mathbb{C}$, we have
(i) for every $w \in U\backslash \bigcup_{j=1}^{n} \operatorname{img}(\gamma_j)$

$$f(w) \sum_{j=1}^{n} I(\gamma_j; w) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{z-w} \, \mathrm{d}z$$

and (ii)

$$\sum_{j=1}^{n} \int_{\gamma_j} f(z) \, \mathrm{d}z = 0.$$

Corollary. Let $U \subset \mathbb{C}$ be open and let β_1, β_2 be closed curves in U such that $I(\beta_1; w) = I(\beta_2; w)$ for all $w \in \mathbb{C}\backslash U$. Then

$$\int_{\beta_1} f(z) \, \mathrm{d}z = \int_{\beta_2} f(z) \, \mathrm{d}z$$

for any holomorphic function $f : U \to \mathbb{C}$.

5.2 Homotopic Curves

We have just answered the question of "for which closed curves in a given domain is the Cauchy theorem valid". The answer is curves that are homologous to zero in the domain. However, this condition may be difficult to check. There is a more restrictive condition, called being null-homotopic, which implies being homologous to zero.

Definition (Homotopic curves). Let $U \subset \mathbb{C}$ be a domain, and let $\gamma_0, \gamma_1 : [a, b] \to U$ be closed curves. We say γ_0 is homotopic to γ_1 in U if there is a continuous map $H : [0, 1] \times [a, b] \to U$ such that

$$H(0, t) = \gamma_0(t) \, \forall t \in [a, b]$$

$$H(1, t) = \gamma_1(t) \, \forall t \in [a, b]$$

$$H(s, a) = H(s, b) \, \forall s \in [0, 1].$$

Such a map H is called a homotopy between γ_0 and γ_1. We can interpret this as saying it is possible to deform one curve into the other continuously while remaining in the domain.

Definition (Null-homotopic). A closed curve $\gamma : [a, b] \to U$ is null-homotopic in U if it is homotopic to a constant curve in U (image equal to one point in U).

Theorem (Null-homotopic implies homologous to zero). If $\gamma_0, \gamma_1 : [a, b] \to U$ are homotopic closed curves in U, then $I(\gamma_0; w) = I(\gamma_1; w)$ for every $w \in \mathbb{C}\backslash U$. In particular, if a closed curve γ in U is null-homotopic in U, then it is homologous to zero in U.

Corollary. If $\gamma_0, \gamma_1 : [a, b] \to U$ are homotopic closed curve in U, then

$$\int_{\gamma_0} f(z) \, \mathrm{d}z = \int_{\gamma_1} f(z) \, \mathrm{d}z$$

for any holomorphic function $f : U \to \mathbb{C}$.

5.3 Simply Connected Condition

Definition (Simply connected). A domain U is simply connected if every closed curve in U is null-homotopic in U.

Theorem (Cauchy's theorem for simply connected domains). If U is simply connected, then

$$\int_{\gamma} f(z)\,\mathrm{d}z = 0$$

for every holomorphic function $f : U \to \mathbb{C}$ and every closed curve γ in U.

It turns out the converse to this theorem is also true, but is harder to prove. Hence U is simply connected if and only if $\int_{\gamma} f(z)\,\mathrm{d}z = 0$ for every closed curve in U and every holomorphic function f on U. This demonstrates an equivalence between simply connectedness (a topological property) and the validity of the Cauchy condition (an analytic condition).

6 Singularities

Begin with some motivation for this section. For a holomorphic function g we may want to find $\int_{\gamma} g(z)\,\mathrm{d}z$ where there are several "bad" points or singularities in the domain.

6.1 Isolated, Removable and Essential Singularities

Definition (Isolated singularity). Let $U \subset \mathbb{C}$ be open. If $a \in U$ and $f : U \backslash \{a\} \to \mathbb{C}$ is holomorphic, then we say f has an isolated singularity at a.

Definition (Removable singularity). An isolated singularity a of f is a removable singularity of f if f can be defined at a so that the extended function is holomorphic on U.

Proposition (Characterising removable singularities). Suppose U is open, $a \in U$ and $f : U\backslash\{a\} \to \mathbb{C}$ is holomorphic. Then the following are equivalent:
(a) f has a removable singularity at a
(b) $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$
(c) there is a disk $D(a, \varepsilon) \subset U$ such that $|f(z)|$ is bounded in $D(a, \varepsilon)\backslash\{a\}$
(d) $\lim_{z\to a}(z - a)f(z) = 0$.

Definition (Pole). If $a \in U$ is an isolated singularity of f, then a is a pole of f if

$$\lim_{z\to a} |f(z)| = \infty.$$

Definition (Essential singularity). If $a \in U$ is an isolated singularity of f, then a is an essential singularity of f if a is neither a removable singularity nor a pole (i.e. $\lim_{z\to a} |f(z)|$ does not exist in $[0, \infty]$).

6.2 Characterising Poles

We have some results that we can use to characterise poles and singularities.

Proposition (Characterising poles). Let $f : U\backslash\{a\} \to \mathbb{C}$ be holomorphic. The following are equivalent:
(a) f has a pole at a
(b) there exist $\varepsilon > 0$ and holomorphic $h : D(a, \varepsilon) \to \mathbb{C}$ with $h(a) = 0$ and $h(z) \neq 0$ for $z \neq a$, such that $f(z) = \frac{1}{h(z)}$ for $z \in D(a, \varepsilon)\backslash\{a\}$

(c) there exists a unique $k \in \mathbb{N}$ and unique holomorphic $g : U \to \mathbb{C}$ with $g(a) \neq 0$ such that $f(z) = (z-a)^{-k} g(z)$ for $z \in U \backslash \{a\}$. This k is the *order* of the pole.

We may remark that from the $(a) \implies (c)$ relation, this means there is no holomorphic function on a punctured disk $f : D(a, R) \backslash \{a\} \to \mathbb{C}$ such that $|f(z)| \to \infty$ as $z \to a$ at the rate of a negative non-integer power of $|z - a|$.

Definition (Meromorphic). Let U be open and $S \subset U$ be a discrete subset of U (so all points in S are isolated points). If $f : U \backslash S \to \mathbb{C}$ is holomorphic and each $a \in S$ is either a removable singularity or a pole of f, then f is a meromorphic function on U.

Theorem (Casorati-Weierstrass Theorem). If $f : U \backslash \{a\} \to \mathbb{C}$ is holomorphic and $a \in U$ is an essential singularity of f, then for any $\varepsilon > 0$, the set $f(D(a, \varepsilon) \backslash \{a\})$ is dense in $\mathbb{C}$.

Theorem (Picard's Theorem). If $f : U \backslash \{a\} \to \mathbb{C}$ is holomorphic and $a \in U$ is an essential singularity of f, then there is $w \in \mathbb{C}$ such that for any $\varepsilon > 0$, $\mathbb{C} \backslash \{w\} \subset f(D(a, \varepsilon) \backslash \{a\})$, i.e. in any neighbourhood $D(a, \varepsilon) \backslash \{a\}$, f attains all complex values except possibly one.

6.3 Laurent Expansions

The Laurent expansion allows us to generalise the concept of a Taylor expansion in a way that makes it easier to work with singularities.

Theorem (Laurent expansion). Let f be holomorphic on an annulus $A = \{z \in \mathbb{C} : r < |z - a| < R\}$ where $0 \leqslant r < R \leqslant \infty$. Then
(i) f has a unique convergent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

where c_n are constants
(ii) for any $\rho \in (r, R)$, the coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z - a)^{n+1}} \, \mathrm{d}z$$

(iii) if $r < \rho' \leqslant \rho < R$ then the (i) series converges uniformly on the set $\{z \in \mathbb{C} : \rho' \leqslant |z - a| \leqslant \rho\}$.

Remark. This theorem shows that if f is holomorphic on an annulus, then f can effectively be written as a sum of holomorphic functions defined on the outer disk and the region outside the inner disk respectively.

6.4 Classifying Singularities

Suppose $f : D(a, R) \backslash \{a\} \to \mathbb{C}$ is holomorphic (so $z = a$ is an isolated singularity of f). Then by Laurent series, there are unique $\{c_n\} \in \mathbb{C}$ with

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - a)^n$$

for $z \in D(a, R) \backslash \{a\}$. We can check that
(i) $c_n = 0$ for all $n < 0 \iff z = a$ is a removable singularity
(ii) $c_{-k} \neq 0$ for sum $k \geqslant 1$, and $c_{-n} = 0$ for all $n \geqslant k + 1 \iff z = a$ is a pole of order k
(iii) $c_n \neq 0$ for infinitely many $n < 0 \iff z = a$ is an essential singularity.

7 The Residue Theorem

7.1 Residue Theorem Statement

Definition (Residue and principal part). Let $f : D(a,R)\backslash\{a\}$ be holomorphic. The coefficient c_{-1} of the Laurent series of f in $D(a,R)\backslash\{a\}$ is called the residue of f at a, denoted $\mathrm{Res}_f(a)$.
The principal part f_p of f at a is defined by

$$f_p := \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}.$$

Theorem (Residue theorem). Let U be an open set, let $\{a_1, a_2, \ldots, a_k\}$ be a finite set, and let $f : U\backslash\{a_1, \ldots, a_k\} \to \mathbb{C}$ be holomorphic. If γ is any closed curve in U homologous to zero in U with $a_j \notin \mathrm{img}(\gamma)$ for all j, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k} I(\gamma; a_j)\mathrm{Res}_f(a_j).$$

7.2 Computing Residues

There are a few useful facts for calculating residues. There are also two lemmas that make it easier to compute line integrals of functions. Jordan's lemma is useful for "integrals on large semicircles", while the other lemma is useful for "integrals on small circular arcs".

Lemma (Jordan's lemma). Let f be a continuous complex-valued function on the semicircle $C_R^+ = \mathrm{img}(\gamma_R^+)$ in the closed upper half-plane $\mathbb{H}$, where $R > 0$ and $\gamma_R^+(t) = Re^{it}$, $0 \leqslant t \leqslant \pi$. Then for $\alpha > 0$

$$\left|\int_{\gamma_R^+} f(z)e^{i\alpha z}\,\mathrm{d}z\right| \leqslant \frac{\pi}{\alpha}\sup_{z\in C_R^+}|f(z)|.$$

In particular if f is continuous in $\mathbb{H}\backslash D(0,R_0)$ and if $\sup_{z\in C_R^+}|f(z)| \to 0$ as $R \to \infty$, then for each $\alpha > 0$,

$$\int_{\gamma_R^+} f(z)e^{i\alpha z}\,\mathrm{d}z \to 0$$

as $R \to \infty$.

Lemma (Integrals on small circular arcs). Let f be holomorphic in $D(a,R)\backslash\{a\}$ with a simple pole at $z = a$. If $\gamma_\varepsilon : [\alpha, \beta] \to \mathbb{C}$ is the circular arc $\gamma_\varepsilon(t) = a + \varepsilon e^{it}$, then

$$\lim_{\varepsilon\to 0^+}\int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = (\beta-\alpha)i\,\mathrm{Res}_f(a).$$

7.3 The Argument Principle

Theorem (Argument principle). Let f be a meromorphic function on a domain U with finitely many zeros at points $a_1, \ldots, a_k$ and finitely many poles at points $b_1, \ldots, b_l$. If γ is a closed curve in U homologous to zero in U with all a_i, b_j not in its image, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = \sum_{i=1}^{k} I(\gamma; a_i)\mathrm{ord}_f(a_i) - \sum_{j=1}^{l} I(\gamma; b_j)\mathrm{ord}_f(b_j).$$

We can interpret $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$ as the number of times the *image* curve $f\circ\gamma$ winds around 0 as we move along γ.

Definition (Curve bounding domain). Let Ω be a domain and let γ be a closed curve in $\mathbb{C}$. We say that γ bounds Ω if $I(\gamma; w) = 1\ \forall w \in \Omega$, and $I(\gamma; w) = 0\ \forall w \in \mathbb{C} \backslash (\Omega \cup \operatorname{img}(\gamma))$.

Corollary (Argument principle for domains bounded by closed curves). Let γ be a closed curve bounding a domain Ω, and let f be meromorphic in an open set U containing $\Omega \cup \operatorname{img}(\gamma)$. Suppose that f has no zeros or poles on $\operatorname{img}(\gamma)$, and precisely N zeros and P poles in Ω, counted with multiplicity. Then N and P are finite, with

$$N - P = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} = I(f \circ \gamma; 0).$$

7.4 Local Degree Theorem and Rouché's Theorem

The local degree theorem relates the order of a zero at a point to the number of roots of an equation in a small disk around that point.

Definition (Local degree). Let f be a holomorphic function on a disk $D(a, R)$ and assume f is non-constant. The local degree of f at a, written $\deg_f(a)$, is the order of the zero of $f(z) - f(a)$ at $z = a$. This is a finite positive integer.

Theorem (Local degree theorem). Let $f : D(a, R) \to \mathbb{C}$ be holomorphic and non-constant with $\deg_f(a) = d > 0$. Then $\exists\ r_0 > 0$ such that for any $r \in (0, r_0]$ there is $\varepsilon > 0$ such that for every w with $0 < |f(a) - w| < \varepsilon$, the equation $f(z) = w$ has precisely d distinct roots in $D(a, r) \backslash \{a\}$.

Corollary (Open mapping theorem). A non-constant holomorphic function on a domain maps open sets to open sets.

Theorem (Rouché's theorem). Let γ be a closed curve bounding a domain Ω, and let f, g be holomorphic functions on an open set U containing $\Omega \cup \operatorname{img}(\gamma)$. If $|f(z) - g(z)| < |g(z)|\ \forall z \in \operatorname{img}(\gamma)$, then f and g have the same number of zeros in Ω, counted with multiplicity.