PQM - Clebsch-Gordan Decomposition

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In Part II Representation Theory, the irreducible representations of SU(2) are denoted by V_n (dimension $n+1$) where $n \in \mathbb{N}$. We showed that tensor products of such V_n decompose as $V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|}$. The same applies in the context of quantum mechanics (spin operators obey the SU(2) Lie algebra), but instead these same representations are indexed by a parameter $l \in \{0, 1/2, 1, 3/2, \dots\}$ where V_l has dimension $2l+1$. Write

$$V_l \otimes V_k \cong V_{l+k} \oplus V_{l+k-1} \oplus \cdots \oplus V_{|l-k|+1} \oplus V_{|l-k|}.$$

Write $\underline{s} = \{|s\rangle, |s-1\rangle, \dots, |-s\rangle\}$, a space of dimension $2s+1$.

For example, a system of three spin-1 particles can be written as $\underline{1} \otimes \underline{1} \otimes \underline{1}$, which decomposes as

$$\begin{aligned}
\underline{1} \otimes \underline{1} \otimes \underline{1} &= \underline{1} \otimes (\underline{0} \oplus \underline{1} \oplus \underline{2}) \\
&= (\underline{1} \otimes \underline{0}) \oplus (\underline{1} \otimes \underline{1}) \oplus (\underline{1} \otimes \underline{2}) \\
&= \underline{1} \oplus (\underline{0} \oplus \underline{1} \oplus \underline{2}) \oplus (\underline{1} \oplus \underline{2} \oplus \underline{3}) \\
&= \underline{0} \oplus \underline{1} \oplus \underline{1} \oplus \underline{1} \oplus \underline{2} \oplus \underline{2} \oplus \underline{3}.
\end{aligned}$$

This shows how many copies of each spin state there are in the combined state, and hence we can use this to find the degeneracies corresponding to spin eigenvalues (e.g. $1(1+1)\hbar = 2\hbar$ appears three times). Note both sides of the equation are states with dimension 27.