Galois Theory - Finding Subfields

Abigail Tan
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Example. Show that $\mathbb{Q}(\zeta_{21})$ has exactly three subfields of degree 6 over $\mathbb{Q}$. Show that one of them is $\mathbb{Q}(\zeta_7)$, one is real, and the other is a cyclic extension $K/\mathbb{Q}(\zeta_3)$.

Solution. Write $\zeta = \zeta_{21}$. Since $\mathbb{Q}(\zeta)$ is a cyclotomic extension, it has Galois group isomorphic to $(\mathbb{Z}/21\mathbb{Z})^\times \cong C_{12}$. Write this as $G = \{1, 2, 4, 5, 8, 10, 11, 13, 16, 17, 19, 20\}$ (where each element i represents σ^i, with $\sigma(\zeta) = \zeta^i$). Subfields of order 6 correspond to groups of order $12/6 = 2$, which are those generated by 8, 13 and 20. In general, to find elements fixed by a subgroup $H \leq G$, we consider $\sum_{\sigma \in H} \sigma(\zeta)$.
For $H = \langle 8 \rangle$, notice $\zeta^3 + \zeta^{24} = \zeta^3 + \zeta^3 = 2\zeta^3$ is fixed by H, and $\zeta^3 = \zeta_7$ has degree 6 over $\mathbb{Q}$, so H corresponds to $\mathbb{Q}(\zeta_7)$.
For $H = \langle 13 \rangle$, $\zeta^3 + \zeta^{39} = \zeta^3 + \zeta^{-3} = \zeta_7 + \zeta_7^{-1} = 2\cos\frac{2\pi}{7}$ (degree 3 over $\mathbb{Q}$), and H also fixes $\zeta_3 = \zeta^7$ (degree 2 over $\mathbb{Q}$) so H corresponds to $\mathbb{Q}(\cos\frac{2\pi}{7}, \zeta_3)$.
For $H = \langle 20 \rangle$, H fixes $\zeta + \zeta^{20} = \zeta + \zeta^{-1}$ giving $\mathbb{Q}(\cos\frac{2\pi}{21})$.