Logic and Set Theory (L1-8)

Abigail Tan
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Propositional Logic

Definition (Valuation). A valuation is a function $v : L \to \{0, 1\}$ such that
(i) $v(\perp) = 0$
(ii) $v(p \Rightarrow q) = \begin{cases} 0 \text{ if } v(p) = 1, v(q) = 0 \\ 1 \text{ otherwise.} \end{cases}$

A valuation is determined by its values on the primitives, and any values will do.

Definition (Semantic entailment). S semantically entails t (write $S \models t$) if every valuation where $v(s) = 1\ \forall s \in S$ has $v(t) = 1$.

Definition (Syntactic entailment axioms).
1. $p \Rightarrow (q \Rightarrow p)$
2. $(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$
3. $(\neg\neg p) \Rightarrow p$

Theorem (Deduction Theorem). Let $S \subset L$, $p, q \in L$. Then $S \vdash (p \Rightarrow q)$ if and only if $S \cup \{p\} \vdash \{q\}$.

Definition (Consistent and deductively closed). S is consistent if $S \nvdash \perp$. S is deductively closed if $S \vdash p$ implies $p \in S$.

Definition (Model). Say v is a model of S if $v(s) = 1$ for all $s \in S$.

Theorem (Model existence lemma). Let $S \subset L$ be consistent. Then S has a model.
A valuation makes half of all sentences true, so expand S to "swallow up" one of p and $\neg p$ for each p.

Theorem (Completeness Theorem for propositional logic). $S \vdash t$ if and only if $S \models t$.
Combine soundness and adequacy theorems.

Theorem (Compactness Theorem). Let $S \subset L$ and $t \subset L$ with $S \models t$. Then some finite $S' \subset S$ has $S' \models t$.
Immediate from replacing $\models$ with $\vdash$ as proofs are finite.
If every finite subset of S has a model, then S has a model.

Theorem (Decidability Theorem). Let $S \subset L$ be finite and $t \in L$. Then there is an algorithm to decide, in finite time, whether or not $S \vdash t$.
Immediate from replacing $\vdash$ with $\models$ by drawing a truth table.

Well-Orderings

Definition (Total order). A total order is a pair $(X, <)$ where X is a set and $<$ is an irreflexive, transitive, and trichotomous relation on X.

Definition (Well-ordering). A total order $(X, <)$ is a well-ordering if every nonempty subset has a least element.

Definition (Initial segment). A subset I of a total order X is an initial segment if $y < x$ implies $y \in I$ for all $x \in I$.

Proposition (Induction). Let X be well-ordered and $S \subset X$ be such that if $y \in S$ for all $y < x$, then $x \in S$. Then $S = X$.

Proposition (Recursion). Let X be a well-ordering and Y any set. Let $G : \mathcal{P}(X \times Y) \to Y$. Then $\exists! f : X \to Y$ such that $f(x) = G(f|I_x)$.

Proposition (Subset collapse). Let X be a well-ordering and $Y \subset X$. Then Y is isomorphic to a unique initial segment $I \subset X$. (In this case, write $Y \leq X$).

Definition (Extending a well-ordering). For well-orderings $(X, <_X)$ and $(Y, <_Y)$, say Y extends X if $X \subset Y$ with $<_Y |_X = <_X$ and X is an initial segment of $(Y, <_Y)$.

Proposition (Well-ordering superset). Let $\{X_i : i \in I\}$ be a nested set of well-orderings. Then there exists a well-ordering X s.t. $X \supseteq X_i$ for all i.

Ordinals

Definition (Ordinal). An ordinal is a well-ordered set where isomorphic ones are considered the same. Any well-ordered set X is isomorphic to a unique ordinal α, the order-type of X. Write $\alpha = \text{ord}(X)$.

Theorem (Well-orderedness of ordinals). Let $\alpha \in \text{Ord}$. Then the collection of all ordinals $\beta < \alpha$ form a well-ordered set I_α. Moreover, $\text{ord}(\alpha) = \alpha$.

It follows that any non-empty set X of ordinals has a least element. Further, the Burali-Forti paradox states that the ordinals do not form a set.

Definition (Successor and limit ordinals). Let $\alpha \in \text{Ord}$. If α has a greatest element, then it is a successor ordinal. Otherwise, it is a limit ordinal.

Proposition (Uncountable ordinal). There exists an uncountable ordinal.
Let W be set of well-orderings of subsets of $\mathbb{N}$ and V be the set of order types of well-orderings in W. Then V is the set of countable ordinals. Letting $\alpha = \text{ord}(V) \in \text{Ord}$, suppose α is countable. Then $\alpha \in V$. But $V \cong \alpha \cong I_\alpha$, a proper initial segment of V (QEA).

Theorem 20 (Hartogs' Lemma). Let X be a set. Then there exists an ordinal α such that there is no injection $\alpha \to X$.