Linear Algebra

Definitions and Results

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1 Vector Spaces

1.1 Quotient Spaces

Let $U \leqslant V$. The quotient space V/U is the abelian group V/U equipped with the scalar multiplication $F \times V/U \to V/U$ with $(\lambda, v+U) \mapsto \lambda v + U$.

1.2 Steinitz Exchange Lemma

Let V be a finite-dimensional vector space. Take $(v_1, \dots, v_m)$ linearly independent and take $(w_1, \dots, w_n)$ spanning V. Then $m \leqslant n$ and up to reordering, $(v_1, \dots, v_m, w_{m+1}, \dots, w_n)$ spans V.

1.3 Direct Sum Equivalence

The following are equivalent conditions for a sum of vector spaces to be a direct sum:
(i) $\sum_{i=1}^k V_i = \bigoplus_{i=1}^k V_i$
(ii) $\forall\ 1 \leqslant i \leqslant k$, $V_i \cap (\sum_{j \leqslant i} V_j) = \{0\}$
(iii) For any basis $\mathcal{B}_i$ of V_i, $\mathcal{B} := \cup \mathcal{B}_i$ is a basis of $\sum_{i=1}^k V_i$.

2 Change of Basis

2.1 Change of Basis Formula

Let $\alpha : V \to W$ be linear. Let $B = \{v_1, \dots, v_n\}$, $B' = \{v_1', \dots, v_n'\}$, $C = \{w_1, \dots, w_m\}$, $C' = \{w_1', \dots, w_m'\}$. We define

$$P = \mathrm{Id}_{B',B} = \big([v_1']_B, \dots, [v_n']_B\big)$$

$$Q = \mathrm{Id}_{C',C} = \big([w_1']_C, \dots, [w_m']_C\big)$$

then if $A = [\alpha]_{B,C}$ and $A' = [\alpha]_{B',C'}$, then $A' = Q^{-1}AP$.

2.2 Rank Lemma

Let V, W be vector spaces of dimension n and m respectively. Let $\alpha : V \to W$ be a linear map. Then there exists a basis B of V and C of W such that

$$[\alpha]_{B,C} = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

where r satisfies $\dim(\ker \alpha) = n - r$. It follows that any $m \times n$ matrix is equivalent to

$$\left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

for some $r = r(\alpha)$, and hence by transitivity two matrices are equivalent if and only if they have the same column (or row) rank.

3 Dual Spaces and Maps

3.1 Dual Spaces

Let V be a vector space over F. Then the dual of V, written V^*, is defined by

$$V^* = L(V, F) = \{\alpha : V \to F \text{ linear}\}.$$

3.2 Dual Basis

Suppose V has a finite basis $B = \{e_1, \ldots, e_n\}$. Then there exists a basis for V^* given by $B^* = \{\varepsilon_1, \ldots, \varepsilon_n\}$ where

$$\varepsilon_j \left(\sum_{i=1}^{n} a_i e_i \right) = a_j$$

for $1 \leqslant j \leqslant n$, so $\varepsilon_j(e_i) = \delta_{ij}$.

3.3 Annihilator

If $U \subseteq V$, the annihilator of U is

$$U^o = \{\alpha \in V^* | \forall u \in U, \alpha(u) = 0\}.$$

3.4 Dual Map

Let V and W be vector spaces with $\alpha : V \to W$ linear. Then $\alpha^* : W^* \to V^*, \varepsilon \mapsto \varepsilon \circ \alpha$ is a linear map called the dual map of α.

3.5 Bilinear Forms

Let U and V be vector spaces. Then $\phi : U \times V \to F$ is a bilinear form if it is linear in both components.

4 Determinant

4.1 Determinant Definition

Let $A \in \mathcal{M}_n(F)$ and let a_{ij} be the components of A. Then

$$\det A = \sum_{\sigma \in S_n} \varepsilon(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \ldots a_{\sigma(n)n}.$$

We can check that the determinant of an upper (or lower) triangular matrix is zero.

4.2 Determinant of Block Triangular Matrices

Let $A \in \mathcal{M}_k(F), B \in \mathcal{M}_l(F), C \in \mathcal{M}_{k,l}(F)$. Let

$$N = \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right)$$

then $\det N = \det A \det B$. An analogous result holds for the determinant of block triangular matrices.

4.3 Adjugate Matrix

Let $A \in \mathcal{M}_n(F)$. For $1 \leqslant i, j \leqslant n$, we define $A_{\hat{i}\hat{j}} \in \mathcal{M}_{n-1}(F)$ by removing the ith row and jth column from A. Then we have

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{\hat{i}\hat{j}}$$

and adj A is defined by $[\operatorname{adj} A]_{ij} = (-1)^{i+j} \det(A_{\hat{j}\hat{i}})$. Then $A^{-1} = \frac{1}{\det A} \operatorname{adj} A$.

4.4 Cramer Rule

Let A be a square $n \times n$ matrix and invertible. Let $b \in F^n$. Then the unique solution to $Ax = b$ is given by

$$x_i = \frac{1}{\det A} \det(A_{\hat{i}b})$$

where $A_{\hat{i}b}$ is found by replacing the ith column of A by b.

4.5 Triangulability

We say α is triangulable if it is similar to an upper triangular matrix. Over $\mathbb{C}$ every matrix is triangulable. For $\alpha \in L(V)$, α is triangulable if and only if $\chi_\alpha(t)$ can be written as a product of linear factors over F: $\chi_\alpha(t) = c \prod_{i=1}^{n} (t - \lambda_i)$.

5 Diagonalisability and Minimal Polynomial

5.1 Sharp Criterion of Diagonalisability

Let V be a finite-dimensional vector space over F. Then α is diagonalisable if and only if there exists a polynomial p which is a product of distinct linear factors and has $p(\alpha) = 0$.
For $\alpha, \beta \in L(V)$ diagonalisable, there exists a basis in which both are diagonal if and only if α and β commute.

5.2 Projection Operators

Let α be an endomorphism of V and suppose α satisfies $p(\alpha) = 0$ for some p a product of distinct linear factors

$$p(t) = \prod_{i=1}^{k} (t - \lambda_i).$$

Define the polynomials

$$q_j(t) = \prod_{i \neq j} \frac{t - \lambda_i}{\lambda_j - \lambda_i}$$

so by definition $q_j(\lambda_i) = \delta_{ij}$. We can check that $q(t) := q_1(t) + \cdots + q_k(t) = 1$ for all t.
Define projection operator $\pi_j = q_j(\alpha)$ which is also an endomorphism of V. Then by construction we have $\sum_{j=1}^{k} \pi_j = \text{Id}$ and $\pi_i \pi_j = 0$ for $i \neq j$.

5.3 Minimal Polynomial Definition

Let V be a finite-dimensional vector space over F. The minimal polynomial m_α of α is the nonzero polynomial with smallest degree such that $m_\alpha(\alpha) = 0$.

5.4 Algebraic and Geometric Multiplicity

Let λ be an eigenvalue of α, let a_λ and g_λ be the algebraic and geometric multiplicities, and let c_λ be the multiplicity of λ as a root of the minimal polynomial. Then $1 \leqslant g_\lambda \leqslant a_\lambda$ and $1 \leqslant c_\lambda \leqslant a_\lambda$.

5.5 Equivalence of Diagonalisability Conditions

The following are equivalent:
(i) α is diagonalisable
(ii) $a_\lambda = g_\lambda$ for all eigenvalues λ of α
(iii) $c_\lambda = 1$ for all eigenvalues λ of α.

6 Jordan Normal Form

6.1 Jordan Normal Form Definition

A Jordan normal form matrix contains Jordan blocks on the diagonal. A Jordan block has the form $J_m(\lambda) = \lambda$ for $m = 1$ and for $m = 2$ we have $J_m(\lambda)$ of the following form.

$$\begin{pmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$

Every complex-valued square matrix is similar to a unique Jordan normal form matrix, up to reordering of Jordan blocks.

6.2 General Eigenspace Decomposition

In V, let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of α. If

$$m_\alpha(t) = (t - \lambda_1)^{c_1} \ldots (t - \lambda_k)^{c_k}$$

for some c_i then

$$V = \bigoplus_{j=1}^{k} V_j$$

where $V_j = \ker((\alpha - \lambda_j \mathrm{Id})^{c_j})$.

6.3 Multiplicity Properties

Let J_m be a Jordan block, then $(J_m - \lambda \mathrm{Id})$ is nilpotent. Then on the JNF matrix, a_λ is the sum of sizes of blocks with eigenvalue λ, g_λ is the number of blocks with eigenvalue λ, and c_λ is the size of the largest block with eigenvalue λ.

7 Bilinear Forms

7.1 Diagonalisation

To diagonalise the bilinear form ϕ i.e. write $P^{\mathrm{T}}[\phi]P = D$, we require $A = A^{\mathrm{T}}$.

7.2 Quadratic Forms

A map $Q : V \to F$ is a quadratic form if $\exists$ a bilinear form $\phi : V \times V \to F$ s.t. $\forall\ v \in V$, $Q(v) = \phi(v, v)$.
If $Q : V \to F$ is a quadratic form, then $\exists!$ a symmetric bilinear form $\phi : V \times V \to F$ s.t. $Q(u) = \phi(u, u)$ $\forall u \in V$ (see the polarisation identity).

Theorem. Let $\phi : V \times V \to F$ be a symmetric bilinear form on finite dimensional V. Then $\exists$ a basis B of V s.t. $[\phi]_B$ is diagonal.

Corollary. For $F = \mathbb{C}$, $\dim V < \infty$ and ϕ a symmetric bilinear form on V: $\exists$ B basis of V s.t.

$$[\phi]_B = \left(\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

where r is the rank of ϕ. It follows that every $n \times n$ symmetric matrix over $\mathbb{C}$ is congruent to a unique matrix of the above form.

Corollary. Let $F = \mathbb{R}$, $\dim V = n$ and $\phi : V \times V \to \mathbb{R}$ be symmetric. Then $\exists$ a basis $B = (v_1, \ldots, v_n)$ s.t.

$$[\phi]_B = \begin{pmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{pmatrix}$$

for some $p, q \geqslant 0$ with $p + q = r(\phi)$. We call $s(\phi) := p - q$ the signature of ϕ.

Theorem (Sylvester's law). Let $F = \mathbb{R}$ and $\dim V = n < \infty$. If a real symmetric bilinear form is represented by the above matrix form in two bases B and B' with p, p' and q, q' respectively, then $p = p'$ and $q = q'$.

8 Sesquilinear Forms

8.1 Definition

Let V and W be vector spaces over $\mathbb{C}$. A sesquilinear form on $V \times W$ is a function $\phi : V \times W \to \mathbb{C}$ s.t.
(i) $\phi(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 \phi(v_1, w) + \lambda_2 \phi(v_2, w)$
(ii) $\phi(v, \lambda_1 w_1 + \lambda_2 w_2) = \overline{\lambda_1} \phi(v, w_1) + \overline{\lambda_2} \phi(v, w_2)$.

8.2 Change of Basis

Let $\phi : V \times W \to \mathbb{C}$ be sesquilinear and let $B = (v_1, \ldots, v_m)$ and $C = (w_1, \ldots, w_n)$ be bases of V and W. Then

$$\phi(v, w) = [v]_B^{\mathrm{T}} \, [\phi]_{B,C} \, \overline{[w]_C}.$$

Let B, B' be bases for V and C, C' be bases for W. With $P = [\mathrm{Id}]_{B',B}$ and $Q = [\mathrm{Id}]_{C',C}$, we have

$$[\phi]_{B',C'} = P^{\mathrm{T}} \, [\phi]_{B,C} \, \overline{Q}.$$

8.3 Hermitian Forms

A sesquilinear form $\phi : V \times V \to \mathbb{C}$ is Hermitian if $\forall u, v \in V$, $\phi(u, v) = \overline{\phi(v, u)}$. A sesquilinear form is Hermitian if and only if for any basis B of V, $[\phi]_B = [\phi]_B^{\mathrm{T}}$.

8.4 Polarisation Identity

A Hermitian form ϕ on a complex vector space V is entirely determined by the associated quadratic form $Q : V \to \mathbb{R}$, $v \to \phi(v, v)$ via

$$\phi(u, v) = \tfrac{1}{4}(Q(u + v) - Q(u - v) + iQ(u + iv) - iQ(u - iv)).$$

8.5 Skew-Symmetric Forms

For $F = \mathbb{R}$, the bilinear form $\phi : V \times V \to \mathbb{R}$ is skew-symmetric if $\phi(u, v) = -\phi(v, u)$ for all $u, v \in V$.

9 Inner Product Spaces

9.1 Definition

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. An inner product on V is a positive definite symmetric (or Hermitian) form ϕ on V.
We can prove standard properties such as the triangle and Cauchy-Schwarz inequalities.

9.2 Parseval's Identity

Let $\dim V = n < \infty$ have an orthonormal basis $(e_1, \ldots, e_n)$. Then

$$\langle u, v\rangle = \sum_{i=1}^{n} \langle u, e_i\rangle \overline{\langle v, e_i\rangle} \text{ and } ||u||^2 = \sum_{i=1}^{n} |\langle u, e_i\rangle|^2.$$

9.3 Gram-Schmidt

Let V be an inner product space. Let $(v_i)_{i\in I}$ be a countable, linearly independent family of vectors in V. Then there exists a family $(e_i)_{i\in I}$ of orthonormal vectors with the same span as $(v_i)_{i\in I}$. (See lecture 21 for the procedure).

10 Self-Adjoint and Unitary Operators

10.1 Orthogonal Complement and Projection

Let V be an inner product space with $V_1, V_2 \leqslant V$. V is the orthogonal direct sum of V_1 and V_2 if
(i) $V = V_1 \oplus V_2$
(ii) $\forall (v_1, v_2) \in V_1 \times V_2$, $\langle v_1, v_2\rangle = 0$.
Then we write $V = V_1 \ominus V_2$.

For $W \leqslant V$, we define the orthogonal of W, $W^\perp$, by

$$W^\perp = \{v \in V | \forall w \in W, \langle v, w\rangle = 0\}.$$

We can check that $V = W \ominus W^\perp$.

10.2 Adjoint Maps

Let V, W be finite dimensional inner product spaces and let $\alpha \in L(V, W)$. Then there is a unique linear map $\alpha^* : W \to V$ s.t. $\forall (v, w) \in V \times W$

$$\langle \alpha(v), w\rangle = \langle v, \alpha^*(w)\rangle.$$

Moreover if B, C are orthonormal bases of V, W respectively, then

$$[\alpha^*]_{C,B} = (\overline{[\alpha]_{B,C}})^{\mathrm{T}}.$$

11 Spectral Theory

11.1 Self-Adjoint Maps

Let $\alpha \in L(V)$ be self-adjoint, i.e. $\alpha = \alpha^*$. Then
(i) α has real eigenvalues
(ii) distinct eigenvectors of α are orthogonal
(iii) V has an orthonormal basis of eigenvectors of α.

This corresponds to the following:

Corollary. Let A be a symmetric or Hermitian matrix over $\mathbb{R}$ or $\mathbb{C}$. Then there is an orthogonal (or unitary) matrix P s.t. $P^{\mathrm{T}}AP$ (or $P^{\dagger}AP$) is diagonal and has real-valued entries.

11.2 Unitary Maps

Let $\alpha \in L(V)$ be unitary, i.e. $\alpha^* = \alpha^{-1}$. Then
(i) all eigenvalues of α lie on the unit circle
(ii) eigenvectors corresponding to different eigenvalues are orthogonal
(iii) if V is a finite dimensional complex inner product space, then V has an orthonormal basis consisting of eigenvectors of α.

This corresponds to the following:

Corollary. Let $\phi : V \times V \to F$ be a symmetric (or Hermitian) form. Then there exists an orthonormal basis of V such that ϕ in this basis is represented by a diagonal matrix.

11.3 Simultaneous Diagonalisation

Let V be a finite dimensional real (or complex) vector space. Let $\phi, \psi : V \times V \to F$ be symmetric (or Hermitian) linear forms. Assume ϕ is positive definite. Then there exists a basis of V with respect to which both ϕ and ψ are diagonal.