Galois Theory - Correspondence

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First, a few important definitions.

Definition (Normal). An extension L/K is normal if L/K is algebraic and for every $x \in L$, $m_{x,K}$ splits into linear factors over L. This is equivalent to saying L contains a splitting field for $m_{x,K}$ for all $x \in L$.

Definition (Separable). A polynomial $f \in K[T]$ is separable if it splits into distinct linear factors in a splitting field.

Write $\mathrm{Hom}_K(L, M)$ for the set of K-homomorphisms $L \to M$.

Theorem (Counting embeddings). Let $L = K(x_1, \dots, x_k)$ be a finite extension of K and M/K any extension. Then

$$|\mathrm{Hom}_K(L, M)| \leq [L : K].$$

Equality holds if and only if all $m_{x_i,K}$ split into linear factors over M, and all x_i are separable over K.

Proof. Let $K_1 = K(x_1)$, $e = |\mathrm{Hom}_K(K_1, M)|$, and $d = \deg_K(x_1) = [K_1 : K]$ and note that e is the number of roots of $m_{x_1,K}$ in M so $e \leq d$.
Let $\sigma : K_1 \to M$ be a K-homomorphism. Inducting on k, apply the inductive hypothesis to L/K_1 to say there are at most $[L : K_1]$ extensions of σ to a homomorphism $L \to M$. Hence

$$|\mathrm{Hom}_K(L, M)| \leq e[L : K_1] \leq d[L : K_1] = [L : K].$$

Equality holds if and only if $e = d$ i.e. $m_{x_1,K}$ has d distinct roots in M. Replacing x_1 by x_i gives the conditions. Conversely if they hold, then $|\mathrm{Hom}_K(K_1, M)| = d$. The conditions still hold over K_1 so by induction on k, each $\sigma : K_1 \to M$ has $[L : K_1]$ extensions to $L \to M$, so equality holds. □